LAPLACE'S INTEGRAL TRANSFORMS

In mathematical physics we frequently encounter pairs of functions of the form

$$g(\alpha) = \int_a^b f(t)\, K(\alpha, t)\, dt$$

connecting a pair of functions $f(t)$ and $g(\alpha)$. The function $g(\alpha)$ is called the integral transform of $f(t)$ by the kernel $K(\alpha, t)$. The integral transforms are useful in mathematical analysis and physical applications.

There are a large no. of different kinds of integral transforms depending on the choice of kernel $K(\alpha, t)$ and the range of integration, they are Fourier, Laplace, Henkel and Mellin's transforms having the transforms of the function $f(t)$ for the kernel's $e^{-i\alpha t}$, $e^{-\alpha t}$, $t J_n(\alpha t)$, $t^{\alpha - 1}$ respectively i.e.

$$g(\alpha) = \int_0^\infty f(t)\, e^{-i\alpha t}\, dt \qquad \text{(Fourier transform)}$$

$$g(\alpha) = \int_0^\infty f(t)\, e^{-\alpha t}\, dt \qquad \text{(Laplace transform)}$$

$$g(\alpha) = \int_0^\infty f(t)\, t\, J_n(\alpha t)\, dt \qquad \text{(Hankel transform)}$$

$$g(\alpha) = \int_0^\infty f(t)\, t^{\alpha - 1}\, dt \qquad \text{(Mellin transform)}$$

* Solving the partial differential eqn with boundary values without finding any general solution.

Laplace Transform

The Laplace transform of a function $F(t)$, denoted by $L\{F(t)\}$ for all $t > 0$. or the Laplace transform of $f(t)$ is the function $f(s)$ which is defined as follows:-

$$f(s) = L\{F(t)\} = \lim_{a \to \infty} \int_0^a e^{-st} F(t)\, dt = \int_0^\infty e^{-st} F(t)\, dt \qquad \text{---(1)}$$

Here the operator L is called the Laplace transformation operator. The eqn (1) is true for all values of s for which the

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improper integral converges. The parameter s may be real or complex number; but generally it is taken to be a real number. Here the function $f(t)$ is multiplied by the Kernel e^{-st} and the result is integrated w.r.t t from $t=0$ to $t=\infty$. It is assumed that $F(s)$ exists.

The operator L can be viewed as a transformation which involves the operation of integration and when applied on $f(t)$ yields a new functions $F(s)$ of a new variable s.

The Laplace transform of a function $F(t)$ exists only if the function satisfies the following conditions.

ii) The func^n $f(t)$ should be of exponential order.

This means that the infinite integral $\int_0^\infty F(t)\,dt$ need not exist i.e. $F(t)$ may diverge exponentially for large values of t; however if there is some const. s_0 s.t.

$$|e^{-s_0 t} F(t)| \leq M, \text{ a positive constant} \quad \text{---(2)}$$

for sufficiently large t, then the function $F(t)$ is said to be of exponential order s_0 as $t \to \infty$.

i) The function $F(t)$ should be an arbitrary piecewise continous func^n in every finite interval and that $F(t) = 0$ for all negative values of t.

Proof of validity of above conditions:-

Let $f(t)$ be piecewise continous in every finite interval and of exponential order s_0 as $t \to \infty$.

Let $t_0 > 0$, then Laplace transform of $F(t)$ may be expressed as

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^{t_0} e^{-st} F(t)\,dt + \int_{t_0}^{\infty} e^{-st} F(t)\,dt$$

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Continuity of $F(t)$ in the finite interval $(0, t_0)$ implies that

$$\int_0^{t_0} e^{-st} F(t)\,dt \quad \text{exists.}$$

Further, if $F(t)$ is of exponential order s_0 as $t \to \infty$ then

$$\lim_{t\to\infty} e^{-s_0 t} F(t) \text{ finite}$$

i.e. $|e^{-s_0 t} F(t)| < M$ for all values of $t \leq t_0$.

i.e. $|F(t)| < M e^{s_0 t}$ for all values of $t \geq t_0$.

Now $\left| \int_{t_0}^{\infty} e^{-st} F(t)\,dt \right| \leq \left| \int_{t_0}^{\infty} e^{-st} F(t) \right| dt \leq \int_{t_0}^{\infty} e^{-st} M e^{s_0 t}\,dt$

$$= M \int_{t_0}^{\infty} e^{-(s-s_0)t}\,dt = \frac{M e^{-(s-s_0)t_0}}{s-s_0} \quad \text{if } s > s_0$$

Thus we have

$$\left| \int_{t_0}^{\infty} e^{-st} F(t)\,dt \right| \leq \frac{M e^{-(s-s_0)t_0}}{s-s_0} \quad \text{if } s > s_0$$

Now $\frac{M e^{-(s-s_0)t_0}}{s-s_0}$ can be made vanishingly small by choosing t_0 sufficiently large. Hence

$$\int_0^{\infty} e^{-st} F(t)\,dt \quad \text{exists for } s > s_0.$$

Hence we conclude that Laplace transform (1) exists only if the function $F(t)$ satisfies the two conditions (i) & (ii).

Properties of Laplace Transform

Laplace transform of Elementary function

Here we consider Laplace transform of some elementary functions $f(t)$. In all cases we assume that $f(t)=0$ for $t<0$.

(1) Laplace transform of unity

$f(t)=1$ for $t>0$.

$$L(1)=\int_0^\infty 1\cdot e^{-st}\,dt = \frac{e^{-st}}{(-s)}\Big|_0^\infty$$

$$= 0+\frac{1}{s} = \frac{1}{s}$$

$$\Rightarrow \boxed{L(1)=\frac{1}{s}}$$

(2) Laplace transform of the constant k:-

$f(t)=k$ for $t>0$.

$$L(k)=\int_0^\infty k\cdot e^{-st}\,dt = k\int_0^\infty e^{-st}\,dt$$

$$= k\cdot\frac{e^{-st}}{-s}\Big|_0^\infty = k\cdot\frac{1}{s}=\frac{k}{s}$$

$$\Rightarrow \boxed{L(k)=\frac{k}{s}}$$

(3) Laplace transform of e^{kt}:-

$f(t)=e^{kt}$ for $t\geq 0$.

$$L[e^{kt}]=\int_0^\infty e^{kt}e^{-st}\,dt$$

$$=\int_0^\infty e^{-(s-k)t}\,dt$$

$$=\frac{e^{-(s-k)t}}{(s-k)}\Big|_0^\infty = \frac{1}{s-k}$$

$$\Rightarrow \boxed{L(e^{kt})=\frac{1}{s-k}}$$

(4) Laplace transform of t^n, i.e. $\Gamma(n+1)=t^n$

$$L(t^n)=\int_0^\infty t^n\cdot e^{-st}\,dt$$

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Let $st = x \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{dx}{s}$

$$\Rightarrow L(t^n) = \int_0^\infty \frac{x^n}{s^n} \cdot e^{-x} \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1} dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1)$$

$$= \frac{1}{s^{n+1}} (n!) \cdot \frac{n!}{s^{n+1}}$$

$$\Rightarrow L(t^n) = \frac{1}{s^{n+1}} = \frac{n!}{s^{n+1}}$$

(5) Laplace transform of $\cos hat$ i.e. $F(t) = \cos hat$

$$L\{F(t)\} = L(\cos hat) = \int_0^\infty \cos hat \, e^{-st} dt$$

$$= \int_0^\infty \left(\frac{e^{at} + e^{-at}}{2}\right) \cdot e^{-st} dt$$

$$\left[\because \cos hat = \frac{e^{at} + e^{-at}}{2}\right]$$

$$= \frac{1}{2} \int_0^\infty \left[e^{-(s-a)t} + e^{-(s+a)t}\right] dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-a)t} dt + \int_0^\infty e^{-(s+a)t} dt\right]$$

$$= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2} \frac{s+a+s-a}{s^2-a^2} = \frac{2s}{2(s^2-a^2)}$$

$$\Rightarrow L(\cos hat) = \frac{s}{s^2-a^2}$$

(6) Laplace transform of $\sin hat$ i.e. $F(t) = \sin hat$

$$L(\sin hat) = \int_0^\infty \sin hat \, e^{-st} dt = \int_0^\infty \frac{e^{at} - e^{-at}}{2} \cdot e^{-st} dt$$

$$\left[\because \sin hat = \frac{e^{at} - e^{-at}}{2}\right]$$

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$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} - e^{-(s+a)t}\right] dt$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{2a}{2(s^2-a^2)} = \frac{a}{s^2-a^2}$$

⑦ Laplace transform of sin at i.e. $F(t) = \sin at$

$$L(\sin at) = \int_0^\infty \sin at\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{(e^{iat} - e^{-iat})}{2i} e^{-st}\, dt \quad \left[\sin at = \frac{e^{iat} - e^{-iat}}{2i}\right]$$

$$= \frac{1}{2i}\int_0^\infty \left[e^{-(s-ia)t} - e^{-(s+ia)t}\right] dt$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ia-s+ia}{s^2-i^2a^2}\right] = \frac{2ia}{2i(s^2+a^2)}$$

$$\Rightarrow L(\sin at) = \frac{a}{s^2+a^2}$$

8) Laplace transform of $\cos at$ i.e. $F(t) = \cos at$

$$L(\cos at) = \int_0^\infty \cos at\, e^{-iat}\, dt = \int_0^\infty \frac{e^{iat} + e^{-iat}}{2} \cdot e^{-st}\, dt$$

$$= \frac{1}{2}\left[e^{-(s-ia)t} + e^{-(s+ia)t}\right] dt$$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

10) $F(t) = t \Rightarrow L(t) = \frac{1}{s^2}\left[t\frac{e^{-st}}{s}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{s}\, dt\right] = \frac{1}{s}\left[\frac{e^{-st}}{-s}\right] = \frac{1}{s^2}$

11) $F(t) = Kt \Rightarrow L(Kt) = \int_0^\infty e^{-st}\, Kt\, dt = K\cdot\int_0^\infty t\, e^{-st}\, dt = \frac{K}{s^2}$

12) $F(t) = e^{-at} \Rightarrow L(e^{-at}) = \frac{1}{s+a}$

09) Laplace transform of t^a

$$L\{t^a\} = \int_0^\infty e^{-st} t^a\, dt = \int_0^\infty e^{-z} \frac{z^a}{s^a} \frac{dz}{s} = \frac{1}{s^{a+1}}\int_0^\infty e^{-z} z^{a+1-1}\, dz$$

$$\Rightarrow L(t^a) = \frac{1}{s^{a+1}}\sqrt{(a+1)} \qquad [\because st = z]$$

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Properties of Laplace transform

① **Linearity property**:- If a_1 and a_2 are constants and the Laplace transform of $F_1(t)$ and $F_2(t)$ are $f_1(s)$ and $f_2(s)$ respectively, the Laplace transform of $a_1F_1(t) + a_2F_2(t)$ is given by $a_1f_1(s) + a_2f_2(s)$ i.e.

$$L\{a_1F_1(t) + a_2F_2(t)\} = a_1L\{F_1(t)\} + a_2L\{F_2(t)\}$$

Proof:- $$L[a_1F_1(t) + a_2F_2(t)] = \int_0^\infty e^{-st}[a_1F_1(t) + a_2F_2(t)]\,dt$$

$$= a_1\int_0^\infty e^{-st}F_1(t)\,dt + a_2\int_0^\infty e^{-st}F_2(t)\,dt$$

$$= a_1L\{F_1(t)\} + a_2L\{F_2(t)\}$$

② **Change of Scale property**:- If $f(s)$ is the Laplace transform of $F(t)$ then the $L\{F(at)\}$ is $\frac{1}{a}f\left(\frac{s}{a}\right)$.

Proof:- $$L\{F(at)\} = \int_0^\infty F(at)\,e^{-st}\,dt$$

$$= \int_0^\infty F(u)\,e^{-(s/a)u}\,\frac{du}{a} \quad \left[\text{Put } at = u \Rightarrow t = \frac{u}{a} \Rightarrow dt = \frac{du}{a}\right]$$

$$= \frac{1}{a}\int_0^\infty F(u)\,e^{-(s/a)u}\,du$$

$$L\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right)$$

As, $$L\{F(t)\} = f(s) = \int_0^\infty e^{-st}F(t)\,dt$$

③ **The First translation or the First shifting theorem** (property):-

If $f(s)$ is the Laplace transform of $F(t)$ then,

$$L\{F(t)e^{at}\} = f(s-a)$$

Proof:- $$L\{e^{at}F(t)\} = \int_0^\infty e^{-st}e^{at}F(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t}F(t)\,dt$$

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Put $s = s-a$

$$L\{e^{at}F(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \boxed{L\{e^{at}F(t)\} = F(s) = F(s-a)}$$

(B) Second translation or shifting property or Heaviside's shifting theorem:-

If $f(s) = L\{F(t)\}$ and $g(t) = \begin{cases} F(t-a) & t>a \\ 0 & 0<t<a \end{cases}$

then $L\{g(t)\} = e^{-as} f(s)$

Proof:- $L\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt$

$$= \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$= 0 + \int_0^\infty e^{-st} F(t-a)\,dt$$

$$= \int_0^\infty e^{-st} F(t-a)\,dt$$

$$= \int_0^\infty e^{-s(u+a)} F(u)\,du \quad [\text{Put } u = t-a]$$

$$= e^{-sa}\int_0^\infty e^{-su} F(u)\,du$$

$$\boxed{L\{g(t)\} = e^{-as} f(s)}$$

Example of 1st shifting theorem,

$$L(e^{at}t^n) = \frac{n!}{(s-a)^{n+1}}$$

$\Rightarrow L(e^{at}t^n) = \int_0^\infty e^{at} t^n e^{-st}\,dt$

$= \int_0^\infty e^{-(s-a)t}\, t^n\,dt$

$(s-a)t = z$

$\Rightarrow$ ~~dss dz~~

$\Rightarrow$

$\Rightarrow \frac{z}{s-a} = t$

$\Rightarrow dt = \frac{dz}{s-a}$

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$$= \int_0^\infty e^{-z} \frac{z^n}{(s-a)^n} \frac{dz}{(s-a)}$$

$$= \frac{1}{s} \int_0^\infty e^{-zt} t^n \, dt$$

$$= \frac{1}{(s-a)^{n+1}} \int_0^\infty e^{-z} z^n \, dz$$

$$= \frac{\sqrt{n+1}}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

* $L(e^{at} \cosh bt) = \frac{s-a}{(s-a)^2 - b^2}$

* $L(e^{at} \sinh bt) = \frac{b}{(s-a)^2 - b^2}$

* $L(e^{at} \sin bt) = \frac{b}{(s-a)^2 + b^2}$

* $L(e^{at} \cos bt) = \frac{s-a}{(s-a)^2 + b^2}$

Examples of 2nd shifting theorem

(1) $L(\cos 2t) = L(2\cos^2 t - 1)$

(1) $L(\cos^2 t) = L\left\{\frac{1}{2}(\cos 2t + 1)\right\}$

$$= \frac{1}{2}\left[L(\cos 2t) + L(1)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right]$$

(2) $L(\sin^2 t) = L\left\{\frac{1}{2}(1 - \cos 2t)\right\}$

$$= \frac{1}{2}\left\{L(1) - L(\cos 2t)\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

(3) $L(\cos^2 at) =$

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Laplace transform of the Derivatives of a function:-

Let $f(t)$ be a continous differentiable function and $\frac{dF(t)}{dt}$ is its first derivative. If $F(t)$ and $\frac{dF(t)}{dt}$ are Laplace transformable, then the Laplace transform of the derivative $\frac{dF(t)}{dt}$ is given by,

$$L\left\{\frac{dF(t)}{dt}\right\} = s\,L\{F(t)\} - F(0) = s f(s) - F(0)$$

where, $F(0)$ is the value of $F(t)$ at $t=0$ and $f(s) = L\{F(t)\}$

Proof:- $$L\left\{\frac{dF(t)}{dt}\right\} = \int_0^\infty \left\{\frac{dF(t)}{dt}\right\} e^{-st}\,dt$$

$$= \left[F(t)\,e^{-st}\Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{dt}\,\frac{dF(t)}{dt}\,dt\right]$$

$$= F(t)\,e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}\,F(t)\,dt$$

$$L\left\{\frac{dF(t)}{dt}\right\} = -F(0) + s\,L\{F(t)\} = s f(s) - F(0)$$

$$\boxed{L\{f'(t)\} = s\,L\{f(t)\} - f(0)} \quad \text{---(1)}$$

Existance Theorem

If $F(t)$ is continous and has derivatives of orders $1, 2, \ldots, n$ which are Laplace transformable,

$$L\left\{\frac{d^n F(t)}{dt^n}\right\} = s^n L\{F(t)\} - \sum_{k=0}^{n-1} F^k(0)\,s^{n-k-1}$$

where $F^k(0) = \left.\frac{d^k F(t)}{dt^k}\right|_{t=0}$

Proof:- We know that,

$$L[f'(t)] = s\,L[f(t)] - f(0) \quad \text{---(1)}$$

Replacing $F(t)$ by $\frac{d^{n-1}}{dt^{n-1}}$ in (1), we get

$$L\left\{\frac{d^n F(t)}{dt^n}\right\} = s\,L\left\{\frac{d^{n-1}F(t)}{dt^{n-1}}\right\} - \left\{\frac{d^{n-1}F(t)}{dt^{n-1}}\right\}_{t=0} \quad \text{---(2)}$$

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Similarly,

$$L\left\{\frac{d^{n-1}F(t)}{dt^{n-1}}\right\} = sL\left\{\frac{d^{n-2}F(t)}{dt^{n-2}}\right\} - \left\{\frac{d^{n-2}F(t)}{dt^{n-2}}\right\}_{t=0}$$

Using eqn (3) in eqn (2),

$$L\left\{\frac{d^n F(t)}{dt^n}\right\} = s^2 L\left\{\frac{d^{n-2}F(t)}{dt^{n-2}}\right\} - s\left\{\frac{d^{n-2}F(t)}{dt^{n-2}}\right\}_{t=0} - \left\{\frac{d^{n-1}F(t)}{dt^{n-1}}\right\}_{t=0} \quad \text{---(3)}$$

$$= s^3 L\left\{\frac{d^{n-3}F(t)}{dt^{n-3}}\right\} - s^2\left\{\frac{d^{n-3}F(t)}{dt^{n-3}}\right\}_{t=0} - s\left\{\frac{d^{n-2}F(t)}{dt^{n-2}}\right\}_{t=0} - \left\{\frac{d^{n-1}F(t)}{dt^{n-1}}\right\}_{t=0}$$

$$= \cdots\cdots$$

$$= s^n L\left\{\frac{d^{n-n}F(t)}{dt^{n-n}}\right\} - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - sF^{n-2}(0) - F^{n-1}(0)$$

$$L\left\{\frac{d^n F(t)}{dt^n}\right\} = s^n L\{F(t)\} - \sum_{k=0}^{n-1} F^k(0)\, s^{n-k-1}$$

where the exponents of F are the order of derivatives of $F(t)$ w.r.t t.

OR

Replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (1)

$$L[f''(t)] = sL[f'(t)] - f'(0)$$

$$= s[sL\{f(t)\} - f(0)] - f'(0)$$

$$= s^2 L\{f(t)\} - sf(0) - f'(0)$$

Similarly, $L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$

$$\therefore L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) + \cdots - f^{n-1}(0)$$

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Q. $F(t) = t^{-1/2}$.

$$L(t^n) = \frac{\sqrt{n+1}}{s^{n+1}}$$

Put $n = -\frac{1}{2} \Rightarrow L(t^{-1/2}) = \frac{\sqrt{(-\frac{1}{2}+1)}}{s^{-\frac{1}{2}+1}} = \frac{\sqrt{1/2}}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

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Q. If $L\left(2\sqrt{\frac{t}{\pi}}\right) = \frac{1}{s^{3/2}}$, show that $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$

Ans) $F(t) = 2\sqrt{\frac{t}{\pi}} \Rightarrow F'(t) = \frac{1}{\sqrt{\pi t}}$ Also $F(0) = 0$

$$L\{F'(t)\} = sL\{F(t) - F(0)\} = s\cdot\frac{1}{s^{3/2}} - 0$$

$$\Rightarrow L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$$

Q. $F(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$.

$\Gamma(n+1) = (n+1)\Gamma(n)$

$$L\left(\frac{\cos\sqrt{t}}{2\sqrt{t}}\right) =$$

Let $\frac{d}{dt}\sin\sqrt{t} = \frac{\cos\sqrt{t}}{2\sqrt{t}} = sL\{\sin\sqrt{t} - F(0)\}$

$$L(\sin\sqrt{t}) = L\left(t^{1/2} - \frac{(t^{1/2})^3}{3!} + \frac{(t^{1/2})^5}{5!} - \cdots\right)$$

$$= b\,\frac{\sqrt{(\frac{1}{2}+1)}}{s^{1/2+1}} - \frac{1}{3!}\,\frac{\sqrt{(\frac{3}{2}+1)}}{s^{3/2+1}} + \cdots$$

$$= \frac{\sqrt{(3/2)}}{s^{3/2}} - \frac{1}{3!}\,\frac{\sqrt{5/2}}{s^{5/2}} + \cdots$$

$$= \frac{\cancel{\frac{3}{2}!}}{s^{3/2}} - \frac{1}{3!}\,\frac{\cancel{(\frac{5}{2})!}}{s^{5/2}} + \cdots$$

$$= \frac{\cancel{\frac{3}{2}\frac{1}{2}\sqrt{(1/2)}}}{s^{3/2}} - \frac{1}{3!}\,\cancel{\frac{3}{2}\sqrt{}}$$

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$$= \frac{\sqrt{\pi}}{2s^{3/2}} - \frac{\sqrt{\pi}}{2^2 s^{5/2}} + \frac{1}{2!}\frac{\sqrt{\pi}}{2^4 s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \frac{1}{2^2 s} + \frac{1}{2!}\left(\frac{1}{2^2 s}\right)^2 - \cdots\right\} \quad \left[e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-(1/4s)} \Rightarrow \frac{d}{dt}(\sin\sqrt{t}) = sL\sin\sqrt{t} - 0$$

$$\Rightarrow L\left(\frac{\cos\sqrt{t}}{2\sqrt{t}}\right) = \frac{\sqrt{\pi}}{2\sqrt{s}} e^{(1/4s)}.$$

Initial and Final Value Theorem

① If $L\{f(t)\} = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s[F(s)]$

② If $L\{f(t)\} = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} [sF(s)]$

Laplace transform of integral of $f(t)$:-

$$\boxed{L\int_0^t f(t)\,dt = \frac{1}{s}F(s)}$$

Proof :- Let $\phi(t) = \int_0^t f(t)\,dt$ and $\phi(0) = 0$

then $\phi'(t) = f(t)$

We know, $L[\phi'(t)] = sL[\phi(t)] - \phi(0)$

$\Rightarrow L[\phi'(t)] = sL[\phi(t)]$

$\Rightarrow L[\phi(t)] = \frac{1}{s}L[\phi'(t)]$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}L[f(t)]$

$\Rightarrow L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}F(s)$

Laplace transform of $t.f(t)$

If $L[F(t)] = f(s)$ then, $L^{\circ}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$.

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Proof :- $\frac{d}{ds}\ L[F(t)] = f(s) = \int_0^\infty e^{-st} F(t)\,dt$ — (1)

Differentiating w.r.t s we get,

$$\frac{d}{ds}[f(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} F(t)\,dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st})\, F(t)\,dt$$

$$= \int_0^\infty (-te^{-st})\, F(t)\,dt$$

$$= \int_0^\infty e^{-st}[-t \cdot f(t)]\,dt$$

$$= L[-tF(t)]$$

$$\Rightarrow L[tf(t)] = (-1)\frac{d}{ds}F(s)$$

Similarly, $L[t^2 F(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$

$$\therefore \boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)}$$

Q. Find the Laplace transform of $t \sin at$.

$$L(t \sin at) = L\left(t \cdot \frac{e^{iat} - e^{-iat}}{2i}\right)$$

$$= \frac{1}{2i}\left[L(te^{iat}) - L(te^{-iat})\right]$$

$$= \frac{1}{2i}\left[-\frac{d}{ds}\frac{1}{s-ia} + \frac{d}{ds}\frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{1}{(s-ia)^2} - \frac{1}{(s+ia)^2}\right]$$

$$= \frac{1}{2i}\left[\frac{4ias}{(s-ia)^2(s+ia)^2}\right] = \frac{2as}{(s^2+a^2)^2}$$

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1. Find $L(t \sin hat)$.

$$L(\sin hat) = \frac{a}{s^2 - a^2}$$

$$L(t \sin hat) = -\frac{d}{ds}\left(\frac{a}{s^2-a^2}\right) = \frac{2as}{(s^2-a^2)^2}$$

Q. $F(t) = t^2 \cos at$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right) = \frac{1\cdot(s^2+a^2) - s(2s)}{(s^2+a^2)}$$

$$= \frac{(s^2+a^2)(\cdot}{} = \frac{2s(s^2-3a^2)}{(s^2+a^2)^2} = \frac{s^2+a^2-2s^2}{(s^2+a^2)^2}$$

$$= \frac{-s^2+a^2}{(s^2+a^2)^2}$$

Q. $F(t) = t^2 e^t \sin 4t$.

$$L(e^t \sin 4t) = \frac{4}{(s-1)^2+16}$$

$$L(t^2 e^t \sin 4t) = -\frac{d}{ds}\left(\frac{4}{(s-1)^2+16}\right) = \frac{4(2s-2)}{(s^2-2s+17)^2}$$

$$L(t^2 e^t \sin 4t) = \frac{8(3s^2-6s+13)}{(s^2-2s+17)}$$

Q. $F(t) = t e^{at} \Rightarrow \frac{1}{(s-a)^2} = L\{t e^{at}\}$

$F(t) = t^3 e^{-2t} = \frac{3!}{(s+2)^4}$ ($\frac{d^n}{ds^n}$ $\frac{n!}{(s-a)^{n+1}}$

$F(t) =$

$$L(t^n e^{at}) = (-1)^n \frac{d^n}{ds^n}\left(\frac{1}{s-a}\right) = \frac{n!}{(s-a)^{n+1}}$$

$n = 3$
$a = -2$

$$\therefore L(t^3 e^{-2t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s-a}\right)$$

$$= \frac{3!}{(s+2)^{3+1}} = \frac{3!}{(s+2)^4}$$

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Q. $F(t) = \frac{\sin at}{t}$ and $F(t) = \frac{\sin t}{t}$

Ans) $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

$$L\{e^{iat}\} = \int_0^\infty e^{-st} e^{iat}\, dt = \int_0^\infty e^{-(s-ia)t}\, dt$$

$$= -\left[\frac{e^{-(s-ia)}}{s-ia}\right]_0^\infty = \frac{1}{s-ia}$$

$$= \frac{s+ia}{(s-ia)(s+ia)}$$

$$\Rightarrow L(\cos at + i \sin at) = \frac{s+ia}{s^2+a^2}$$

$$\Rightarrow L(\cos at) + i(\sin at) = \frac{s+ia}{s^2+a^2}$$

Using Equating real and imaginary parts we get,

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

Now we have, $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\, ds$

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty f(s)\, ds = \int_s^\infty f(x)\, dx$$

$$= \int_s^\infty L\{\sin at\}\, dx$$

$$= \int_s^\infty \frac{a\, dx}{a^2+x^2}$$

$$= \left[\tan^{-1} x/a\right]_{x=s}^\infty = \tan^{-1}\infty - \tan^{-1} s/a$$

$$= \pi/2 - \tan^{-1} s/a$$

$$\therefore L\left\{\frac{\sin at}{t}\right\} = \cot^{-1} s/a$$

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Laplace transform of $\frac{F(t)}{t}$:-

If $f(s) = L\{F(t)\}$

then, $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx$

Proof:- Let $f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$\Rightarrow \int_s^\infty f(s)\,ds = \int_s^\infty ds \int_0^\infty e^{-st} F(t)\,dt$$

(As s & t are independent variables the order of integration may be interchanged)

$$\Rightarrow \int_s^\infty f(s)\,ds = \int_0^\infty dt \int_s^\infty e^{-st} F(t)\,ds$$

$$= \int_0^\infty F(t)\,dt \left.\frac{e^{-st}}{-t}\right|_s^\infty$$

$$= \int_0^\infty e^{-st} \frac{F(t)}{t}\,dt$$

$$= \int_0^\infty e^{-st} \frac{F(t)}{t}\,dt$$

$$\Rightarrow \int_s^\infty f(s)\,ds = L\left[\frac{F(t)}{t}\right]$$

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Unit Step Function:-

With the help of unit step functions, we can find the inverse transform of functions, which cannot be determined with previous methods.

The unit step function $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0, & \text{when } t < a \\ 1, & \text{when } t \geq a \end{cases}$$

Then Laplace transform of unit step function is given by,

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st}\, u(t-a)\, dt + \int_a^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a e^{-st} \cdot 0\, dt + \int_a^\infty e^{-st}\, dt$$

$$= 0 + \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$\boxed{L\{u(t-a)\} = \frac{e^{-sa}}{s}}$$

Ex:- Express the following function in terms of unit step functions and find its Laplace transform:

$$f(t) = \begin{cases} 8, & t < 2 \\ 6, & t \geq 2 \end{cases}$$

Soln:- $f(t) = \begin{cases} 8+0, & t<2 \\ 8-2, & t \geq 2 \end{cases} = 8 + \begin{cases} 0, & t<2 \\ -2, & t \geq 2 \end{cases}$

$$= 8 + (-2)\begin{cases} 0, & t<2 \\ 1, & t \geq 2 \end{cases}$$

$$= 8 - 2u(t-2)$$

$\therefore L\{f(t)\} = 8L(1) - 2u(t-2)$

$$= \frac{8}{s} - 2\frac{e^{-s2}}{s}$$

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Second shifting theorem

If $L[f(t)] = F(s)$ then $L[f(t-a)\cdot u(t-a)] = e^{-as} F(s)$

Proof:- $L[f(t-a)\cdot u(t-a)] = \int_0^\infty e^{-st}[f(t-a)\,u(t-a)]\,dt$

$= \int_0^a e^{-st} f(t-a)\cdot 0\,dt + \int_a^\infty e^{-st} f(t-a)(1)\,dt$

$= \int_a^\infty e^{-st} f(t-a)\,dt$

$= \int_0^\infty e^{-s(u+a)} f(u)\,du$ where, $u = t-a$

$= e^{-sa}\int_0^\infty e^{-su} f(u)\,du$

$= e^{-sa} F(s)$

Ex:- Express the following function in terms of unit step function and find its Laplace transform:

$$f(t) = \begin{cases} E, & a<t<b \\ 0, & t \geq b \end{cases}$$

Soln:- $f(t) = E\begin{cases} 1, & a<t<b \\ 0, & t \geq b \end{cases}$ $\quad [L[f(t-a)\cdot u(t-a)] = e^{-as}F(s)]$

$\Rightarrow L\{f(t)\} = E\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$

Ex.2:- Express the following function in terms of unit step function:

$$f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$$

and find it's Laplace transform.

Soln:- $f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$

$= (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$

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$(t-1)u(t-1) - (t-1)u(t-2) + (3-t)u(t-2) + (t-3)u(t-3)$

Ans $= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$

$= e^{-s} L(t) - 2e^{-2s} L(t) + e^{-3s} L(t)$

$= \frac{e^{-s}}{s^2} - 2\frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$

Q. 1) $F(t) = \frac{e^{-at} - e^{-bt}}{t}$ 2) $F(t) = \frac{\cos at - \cos bt}{t}$

3) $F(t) = \frac{\sinh t}{t}$ 4) $F(t) = \frac{\sin^2 t}{t}$

5) $F(t) = \frac{e^{-t} - e^{-3t}}{t}$

2) $\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$ 3) $-\frac{1}{2}\log\frac{s-1}{s+1} = \frac{1}{2}\log\frac{s+1}{s-1}$

4) $\frac{1}{4}\log\frac{s^2+4}{s^2}$ (5) $\frac{1}{2}\log\frac{s^2+b^2}{(s-a)^2}$

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Periodic Functions: (Laplace transform of periodic functions)

Let's consider a periodic function $F(t)$ of period T; so that $F(t+T) = F(t)$; $T>0$. Then the Laplace transform of $F(t)$ is given by,

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^T e^{-st} F(t)\,dt + \int_T^{2T} e^{-st} F(t)\,dt + \cdots + \int_{nT}^{(n+1)T} e^{-st} F(t)\,dt + \cdots$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} F(t)\,dt$$

Substituting $t = x + nT$ i.e. $dt = dx$; we get

$$L\{F(t)\} = \sum_{n=0}^{\infty} \int_0^T e^{-s(x+nT)} F(x+nT)\,dx$$

$$= \sum_{n=0}^{\infty} \int_0^T e^{-sx} e^{-nsT} F(x)\,dx \quad [\text{since } F(x+nT) = F(x) \text{ for } n=0,1,2,3\cdots]$$

$$= \sum_{n=0}^{\infty} e^{-nsT} \int_0^T e^{-sx} F(x)\,dx \quad [\text{since } 1 + e^{-sT} + e^{-2sT} + \cdots = \frac{1}{1-e^{-sT}}]$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-sx} F(x)\,dx$$

Thus, $$\boxed{L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt}$$

Or

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt + \cdots$$

$$= \int_0^T e^{-st}$$

Substituting $t = u + T$ in second integral and $t = u + 2T$ in 3rd integral and so on...

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$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du$$

$$+ \int_0^T e^{-s(u+2T)} f(u+2T)\,du + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du$$

$$+ \cdots$$

$$[f(u) = f(u+T) = f(u+2T) = f(u+3T) = \cdots]$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t)\,dt + e^{-2sT}\int_0^T e^{-st} f(t)\,dt + \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right]$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Ex.1 :- Find the Laplace transform of the waveform

$$f(t) = \left(\frac{2t}{3}\right), \quad 0 \le t \le 3$$

Soln :- $L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$ $\left[f(t) = \frac{2t}{3} \text{ with period } T = T = 3\right]$

$$L\left(\frac{2t}{3}\right) = \frac{1}{1-e^{-3s}} \int_0^3 e^{-st} \left(\frac{2t}{3}\right) dt$$

$$= \frac{1}{1-e^{-3s}} \cdot \frac{2}{3} \left[\frac{te^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^3$$

$$= \frac{2}{3} \cdot \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{2}{3} \frac{1}{1-e^{-3s}} \left[\frac{3e^{-3s}}{-s} + \frac{1-e^{-3s}}{s^2}\right]$$

$$= \frac{2e^{-3s}}{-s(1-e^{-3s})} + \frac{2}{3s^2}$$

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Find the Laplace transform of a square wave,

$f(t) = a$ for $0 < t < T/2$

$= 0$ for $T/2 < t < T$

and $f(t+T) = f(t)$ [Period = T]

Ans. $L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$

$= \frac{1}{1-e^{-sT}} \left[\int_0^{T/2} e^{-st} \cdot a\,dt + \int_{T/2}^{T} e^{-st} \cdot 0\,dt \right]$

$= \frac{1}{1-e^{-sT}} \left[a \cdot \frac{e^{-st}}{-s} \right]_0^{T/2}$

$= \frac{a}{1-e^{-sT}} \left[\frac{e^{-sT/2}}{-s} + \frac{1}{s} \right]$

$= \frac{a(1-e^{-sT/2})}{s(1-e^{-sT})}$

Q.3. Draw the graph of the periodic function

$$f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$$

and find its Laplace transform.

Soln:- Period $= 2\pi = T$

$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$

$= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$

$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \cdot t\,dt + \int_{\pi}^{2\pi} e^{-st} (\pi - t)\,dt \right]$

$= \frac{1}{1-e^{-2\pi s}} \left[\left[\frac{t e^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{(-s)^2} \right]_0^{\pi} + \left[(\pi - t) \frac{e^{-st}}{(-s)} - (-1) \frac{e^{-st}}{(-s)^2} \right]_{\pi}^{2\pi} \right]$

$= \frac{1}{1-e^{-2\pi s}} \left[\left[\frac{\pi e^{-\pi s}}{-s} - \frac{e^{-\pi s}}{(-s)^2} - 0 + \frac{1}{s^2} \right] \right.$

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$$+\left[(\pi-2\pi)\frac{e^{-2\pi s}}{-s}+\frac{e^{-2\pi s}}{s^2}-(\pi-\pi)\frac{e^{-\pi s}}{-s}-\frac{e^{-\pi s}}{s^2}\right]$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{-\frac{\pi e^{-\pi s}}{s}-\frac{e^{-\pi s}}{s^2}+\frac{1}{s^2}+\frac{\pi e^{-2\pi s}}{s}+\frac{e^{-2\pi s}}{s^2}-0-\frac{e^{-\pi s}}{s^2}\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left\{\frac{-\pi}{s}e^{-\pi s}+\frac{\pi}{s}e^{-2\pi s}+\frac{1}{s^2}-\frac{1}{s^2}e^{-\pi s}+\frac{1}{s^2}e^{-2\pi s}-\frac{e^{-\pi s}}{s^2}\right\}$$

$$=\frac{1}{1-e^{-2\pi s}}\left[\frac{\pi}{s}(e^{-2\pi s}-e^{-\pi s})+\frac{1}{s^2}(1+e^{-2\pi s}-2e^{-\pi s})\right]$$

$$=\frac{-\pi s e^{-\pi s}(1-e^{-\pi s})+(1-e^{-\pi s})^2}{s^2(1+e^{-\pi s})(1-e^{-\pi s})}$$

$$=\frac{-\pi s e^{-\pi s}+1-e^{-\pi s}}{s^2(1+e^{-\pi s})}$$

Q. Find the Laplace transform of the function (Half-wave rectifier)

$$f(t)=\begin{cases}\sin\omega t, & 0<t<\pi/\omega\\ 0, & \pi/\omega<t<2\pi/\omega\end{cases}$$

Ans) Period $= T = 2\pi/\omega$

$$L[f(t)]=\frac{1}{1-e^{-s(2\pi/\omega)}}\int_0^{2\pi/\omega}e^{-st}f(t)\,dt$$

$$=\frac{1}{1-e^{-2\pi s/\omega}}\left[\int_0^{\pi/\omega}e^{-st}\sin\omega t\,dt+\int_{\pi/\omega}^{2\pi/\omega}e^{-st}\cdot 0\,dt\right]$$

$$=\frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}(-s\sin\omega t-\omega\cos\omega t)}{(-s)^2+\omega^2}\right]_0^{\pi/\omega}$$

$$=\frac{1}{1-e^{-2\pi s/\omega}}\left[\int e^{ax}\sin bx\,dx=\frac{e^{ax}(a\sin bx-b\cos bx)}{a^2+b^2}\right]$$

Here $a=-s$, $b=\omega$

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$$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{\{e^{-s\pi/\omega}(-s\sin\omega\cdot\pi/\omega + \omega\cos\omega\pi/\omega) - e^{0}(-s\sin 0 + \omega\cos 0)\}}{\omega^2+s^2} \right]$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{e^{-s\pi/\omega}(+\omega) + \omega}{\omega^2+s^2} \right]$$

$$= \frac{\omega(1+e^{-s\pi/\omega})}{(1+e^{-s\pi/\omega})(1-e^{-s\pi/\omega})(\omega^2+s^2)}$$

$$= \frac{\omega}{(\omega^2+s^2)(1-e^{-s\pi/\omega})}$$

Q. Find the Laplace transform of the triangular wave function of period $2c$ given by,

$$f(t) = \begin{cases} t, & 0<t<c \\ 2c-t, & c<t<2c \end{cases}$$

Q. $f(t) = \begin{cases} t, & 0<t<a \\ 2a-t, & a<t<2a \end{cases}$

Q. $f(t) = \begin{cases} t, & 0<t<1/2 \\ 0, & 1/2<t<2 \end{cases}$

Q. $f(t) = \begin{cases} \cos\omega t, & 0<t<\pi/\omega \\ 0, & \pi/\omega<t<2\pi/\omega \end{cases}$ Ans: $\frac{s}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

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1.1 Step Function

The unit step function $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0, & \text{when } t<a \\ 1, & \text{when } t \geq a \end{cases} \quad \text{where } a \geq 0$$

Laplace transform of unit step function

$$L[u(t-a)] = \frac{e^{-as}}{s}$$

$$\Rightarrow L[u(t-a)] = \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$$= \int_0^a 0 \cdot e^{-st}\, dt + \int_a^\infty 1 \cdot e^{-st}\, dt$$

$$= \frac{e^{-st}}{-s}\Big|_a^\infty = \frac{e^{-as}}{s}$$

Q. $f(t) = \begin{cases} E & a<t<b \\ 0, & t \geq b \end{cases}$

$$= E \begin{cases} 1, & a<t<b \\ 0, & t \geq b \end{cases}$$

Ans) $L\{f(t)\} = E\left[\frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\right]$

Q. $f(t) = \begin{cases} 8, & t<2 \\ 6, & t \geq 2 \end{cases}$

$$= \begin{cases} 8+0, & t<2 \\ 8-2, & t \geq 2 \end{cases} \quad = 8 \begin{cases} 0, & t<2 \\ -2, & t \geq 2 \end{cases}$$

$$= 8 + (-2) \begin{cases} 0, & t<2 \\ 1, & t \geq 2 \end{cases}$$

$$= 8 - 2u(t-2)$$

$$L\{f(t)\} = 8L(1) - 2L\{u(t-2)\}$$

$$= 8/s - 2\frac{e^{-2s}}{s}$$

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Second Shifting theorem

Q If $L[F(t)] = f(s)$ then $L[f(t-a)\cdot u(t-a)] = e^{-as} f(s)$

Q. Express the following function in terms of unit step function and find its Laplace transform.

$$f(t) = \begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$$

Ans)
$$f(t) = \begin{cases} t-1; & 1<t<2 \\ 3-t; & 2<t<3 \end{cases}$$

$$= (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)u(t-1) - (t-1)u(t-2) + (3-t)u(t-2) - (3-t)u(t-3)$$

$$= (t-1)u(t-1) - u(t-2)[t-1-3+t] + (t-3)u(t-3)$$

$$= (t-1)u(t-1) - u(t-2)(2t-4) + (t-3)u(t-3)$$

$$= (t-1)u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

$$= e^{-1\cdot s} L(t) - 2e^{-2s} L(t) + e^{-3s} L(t)$$

$$= \frac{e^{-s}}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

Q. $F(t) = \begin{cases} \sin(t-\pi/3), & t > \pi/3 \\ 0, & t < \pi/3 \end{cases}$

$$L\{F(t)\} = e^{-s\pi/3} L(\sin t)$$

$$= e^{-s\pi/3} \cdot \frac{1}{s^2+1}$$

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Theorem :- $L[f(t)u(t-a)] = e^{-as}[f(t-a)]$

Q. $t^2 u(t-3)$

$L[f(t)] = L[t^2 u(t-3)]$

Q. Find the Laplace transform of $e^{-2t}u_\pi(t)$ where,

$$u_\pi(t) = \begin{cases} 0, & t<\pi \\ 1, & t>\pi \end{cases}$$

$u_\pi(t) = u(t-\pi)$

$$L[u_\pi(t)] = L[u(t-\pi)] = \frac{e^{-\pi s}}{s}$$

$$\Rightarrow L[e^{-2\pi}u_\pi(t)] = \frac{e^{-\pi(s+2)}}{s+2}$$

Q. $f(t) = \begin{cases} 0, & 0<t<1 \\ t-1, & 1<t<2 \\ 1, & 2<t \end{cases}$

$$= (t-1)[u(t-1) - u(t-2)] + u(t-2)$$

$$= (t-1)u(t-1) - u(t-2)(t-1) + u(t-2)$$

$$= (t-1)u(t-1) - u(t-2)[t-1-1]$$

$$= (t-1)u(t-1) - (t-2)u(t-2)$$

$$L[f(t)] = L(t-1)u(t-1) - L(t-2)u(t-2)$$

$$= \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$$

Q. $f(t) = \begin{cases} \sin 2t & 2\pi < t < 4\pi \\ 0 & \text{otherwise} \end{cases}$

$$L[f(t)] = L[\sin 2t \cdot u(t-2\pi)] - L[\sin 2t \cdot u(t-4\pi)]$$

$$= e^{-2\pi s} L \sin 2(t+2\pi) - e^{-4\pi s} L \sin 2(t+4\pi)$$

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$= e^{-2\pi s} L \sin 2t - e^{-4\pi s} L(\sin 2t)$

$= e^{-2\pi s} \frac{2}{s^2+4} - e^{-4\pi s} \frac{2}{s^2+4}$

$F(s) = (e^{-2\pi s} - e^{-4\pi s}) \frac{2}{s^2+4}$

Q. $f(t) = \begin{cases} t, & 0<t<2 \\ 0, & 2<t \end{cases}$ $\rightarrow \frac{1-(2s+1)e^{-2s}}{s^2}$

Q. $f(t) = \begin{cases} \sin t & 0<t<\pi \\ t & t>\pi \end{cases}$ $\rightarrow \frac{1+e^{-\pi s}}{s^2+1} + e^{-\pi s}$

Q. $f(t) = \begin{cases} 4 & 0<t<1 \\ -2 & 0<t<3 \\ 5 & t>3 \end{cases}$ $\rightarrow \frac{4-6e^{-s}+7e^{-3s}}{s}$

Q. $e^t u(t-1) = f(t)$ Q. $t^2 u(t-2) = f(t)$

Q. $L[t^2 u(t-3)] = e^{-3s} L(t+3)^2$

$= e^{-3s} L[t^2+6t+9]$

$= e^{-3s}[2/s^3 + 6/s^2 + 9/s]$

Q. $f(t) = \sin t\, u(t-4)$ $\rightarrow \frac{e^{-4s}}{s^2+1}[\cos 4 + s \sin 4]$

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Laplace transform of Dirac-delta function

The funcn $\delta(x)$ which is zero everywhere except at $x=0$ and tends to ∞ in such a manner that.

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

Now one of the property of Dirac-delta function

$$\int_0^{\infty} f(t)\,\delta(t-a)\,dt = f(a)$$

$$\Rightarrow \int_0^{\infty} e^{-st}\,\delta(t-a)\,dt = f(a) = [e^{-st}]_{t=a} = e^{-sa}$$

$$\Rightarrow \boxed{L[\delta(t-a)] = e^{-sa}}$$

If $a=0$. then $L[\delta(t)] = 1$.

Ex- $L[t^3\delta(t-4)]$. $a=4$,

$$= \int_0^{\infty} e^{-st}\,t^3\,\delta(t-4)\,dt = 4^3 e^{-4s}$$

Ex- $L[e^{-4t}\delta(t-3)]$ $a=3$,

$$= \int_0^{\infty} e^{-st} e^{-4t}\,\delta(t-3)\,dt = e^{-12} e^{-3s} = e^{-3(s+4)}$$

Ex- $L\left[\frac{\delta(t-4)}{4}\right] = \frac{e^{-4s}}{4}$

Ex- $L[\cos t \log t\, \delta(t-\pi)] = \cos\pi \log\pi\, e^{-\pi s}$

$$= -\log\pi\, e^{-\pi s}.$$

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Fourier Transformation

The fourier integral transform $F(s)$ of a function $f(t)$ with the kernel $K(s,x)$ is defined as,

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$\Rightarrow f(x) = F^{-1}[F(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, ds$$

List of Formulae for Fourier Integrals:-

1. Fourier integral for $f(x)$ is, $f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos u(t-x)\, du\, dt$

2. Fourier sine integral for $f(x)$ is, $f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \sin ut \sin ux\, du\, dt$

3. Fourier cosine integral for $f(x)$ is,
$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} f(t) \cos ut \cos ux\, du\, dt$$

Fourier complex integrals :- $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iux}\, du \int_{-\infty}^{\infty} f(t)\, e^{iut}\, dt$

Some useful integrals

1. $\int_0^{\infty} \frac{\sin ax}{x}\, dx = \frac{\pi}{2}$
2. $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$
3. $\int_{-\infty}^{\infty} \frac{\sin mx}{(x-b)^2 + a^2}\, dx = \frac{\pi}{a} e^{-am} \sin bm$

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Laplace transform of $\frac{f(t)}{t}$ (i.e. divison by t)

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$$

2. $f(t) = \frac{\sin 3t}{t}$

$$L(\sin 3t) = \frac{3}{s^2+9}$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \frac{3}{s^2+9}\,ds = \int_s^\infty \frac{3}{s^2+3^2}\,ds$$

$$\left[\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}x/a + c\right]$$

$$= 3\left[\frac{1}{3}\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{3}{3}\left[\tan^{-1}\infty - \tan^{-1}\frac{s}{3}\right]$$

$$= \left[\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right]$$

$$= \cot^{-1}\frac{s}{3}$$

$$\int_0^t \frac{\sin 3t}{t}\,dt = \frac{1}{s}\cot^{-1}\frac{s}{3}$$

Q. $f(t) = \frac{1-e^t}{t}$

$$L\left(\frac{1-e^t}{t}\right) = \int_s^\infty L(1-e^t)\,ds = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right)ds$$

$$L(1-e^t) = \frac{1}{s} - \frac{1}{s-1}$$

$$= \left[\log s - \log(s-1)\right]_s^\infty$$

$$= \log\frac{s}{s-1}\Big|_s^\infty$$

$$= \log\frac{1}{1-1/s}\Big|_s^\infty$$

$$= \log 1 - \log\frac{s}{s-1}$$

$$= -\log\frac{s}{s-1} = \log\frac{s-1}{s}$$

Q. $\frac{e^{-at}-e^{-bt}}{t} = \log\frac{s+a}{s+b}$

Q. $F(t) = \frac{1}{t}\sin^2 t = \frac{1}{4}\log\frac{s^2+4}{s^2}$

Q. $F(t) = L^{-1}(\sin ht) = \frac{1}{2}\log\frac{s+1}{s-1}$

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Q. $F(t) = \frac{1-\cos at}{t}$

$L[F(t)] = \int_s^\infty L\left[\frac{1-\cos at}{t}\right] ds$

$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds$

$= \left[\log s - \frac{1}{2}\log(s^2+a^2)\right]_s^\infty$

$s^2+a^2 = t$

$\Rightarrow 2s\,ds = dt$

$= \left[\frac{2\log s - \log(s^2+a^2)}{2}\right]_s^\infty$

$= \left[\frac{\log s^2 - \log(s^2+a^2)}{2}\right]_s^\infty$

$= \frac{1}{2}\left[\log \frac{s^2}{s^2+a^2}\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log \frac{s^2}{s^2+a^2}\right]$

$= -\frac{1}{2}\log \frac{s^2}{s^2+a^2} = \frac{1}{2}\log \frac{s^2+a^2}{s^2}$

Q. $F(t) = \frac{e^{at} - \cos bt}{t}$

$L\left(\frac{e^{at} - \cos bt}{t}\right) = \int_s^\infty [L(e^{at}) - L(\cos bt)]\, ds$

$= \int_s^\infty \left[\frac{1}{s-a} - \frac{s}{s^2+b^2}\right] ds$

$= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\left[\log(s-a)^2 - \log(s^2+b^2)\right]_s^\infty$

$= \frac{1}{2}\log \frac{(s-a)^2}{s^2+b^2} = \frac{1}{2}\left[\log \frac{(1-a/s)^2}{1+b^2/s^2}\right]_s^\infty$

$= \frac{1}{2}\left[\log 1 - \log \frac{(s-a)^2}{s^2+b^2}\right] = \frac{1}{2}\left[\log \frac{s^2+b^2}{(s-a)^2}\right]$

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